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How many 7×7 Latin squares can be partitioned into Youden squares?

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Abstract

Youden (1940) gave examples of Latin squares partitioned into Youden squares. However, the literature seems to have been silent on how many Latin squares of appropriate sizes can be partitioned in this way. We now show that, of the 564 isotopy classes (transformation sets) of 7×7 Latin squares, 29 consist of squares that can be partitioned into 3×7 and 4×7 Youden squares; these 29 isotopy classes belong to 19 of the 147 main classes (species) of 7×7 Latin squares. We also give some related results on partitioning 7×7 Graeco-Latin squares.

1. Introduction

A Youden square is a rectangular arrangement consisting of a set of rows from an $n \times n$ Latin square, the sets of symbols (letters) in the columns of the rectangle being the n blocks of a symmetric balanced incomplete block design. A review of Youden squares was provided by Preece [5].

Youden [7] gave examples of Latin squares partitioned into Youden squares. His 7×7 example, with partitioning into a 3×7 and a 4×7 Youden square, is the following:

$$
\begin{array}{ccccccc}
A & B & C & D & E & F & G \\
B & C & D & E & F & G & A \\
D & E & F & G & A & B & C \\
\hdashline
C & D & E & F & G & A & B \\
E & F & G & A & B & C & D \\
F & G & A & B & C & D & E \\
G & A & B & C & D & E & F
\end{array}
.
$$

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Youden's examples invite the question 'How many Latin squares can be so partitioned?', but answers seem not to have been known. We now provide an answer for 7×7 Latin squares. Knowledge of this answer is of potential statistical use in orchard experimentation, where row-and-column designs such as Latin squares are often used for allocating treatments to trees planted on a rectangular grid but where the later years of an experiment or series of experiments may make use of only a subset of the original rows of a design. There is, however, a wider combinatorial interest in the embedding of Youden squares in larger combinatorial structures. For example, 3×7 Youden squares are embedded in each of the mutually orthogonal 10×10 Latin squares of Parker [3], whereas Owens and Preece [2] produced an 11×11 Latin square with the properties (i) it is not based on a group even though the permutation between any two rows is an 11-cycle, and (ii) it can be partitioned into a 5×11 Youden square and a 6×11 Youden square.

2. Enumeration of partitionable 7 × 7 Latin squares

Norton [1] and Sade [6] classified 7×7 Latin squares into 564 'isotopy classes' (also known as 'transformation sets') which can be grouped into 147 'main classes' (alias 'species'). Preece [4] similarly classified 4×7 Youden squares into 6 isotopy classes, each also constituting a main class, and showed that there is but a single main class of 3×7 Youden squares. These results (details of which should be sought in the original papers) can readily be used to enumerate and identify the isotopy classes of 7×7 Latin squares partitionable into Youden squares.

Members of all such isotopy classes can be obtained by taking a representative member of each of the 6 main classes of 4×7 Youden squares and then completing this representative as a Latin square in all possible ways. The representative members can be taken to be those that were given by Preece [4] and that are reproduced here in Table 1. Each of these members can be completed as a Latin square by adjoining a 3×7 Youden square comprising any one of the 8 sets of rows given in Table 2. Any pair of rows within any one of these 8 arrangements has the property that no subset of the

Table 1
Representative members of the 6 isotopy classes of 4×7 Youden squares

1.	*A B C D E F G* *B C D E F G A* *C D E F G A B* *E F G A B C D*	2.	*A B E F G C D* *B D C A E F G* *C F D E B G A* *E C G D F A B*
3.	*A F E D B C G* *B D C E G F A* *C B G F E A D* *E C D A F G B*	4.	*A F E D B C G* *B C G F E A D* *C B D E G F A* *E D C A F G B*
5.	*A F E D B C G* *B C D E F G A* *C B G F E A D* *E D C A G F B*	6.	*A F E D B C G* *B C D E G F A* *C B G F E A D* *E D C A F G B*

Table 2
Sets of rows that can be used to complete the Youden squares of Table 1 as Latin squares

X. *D E F G A B C* *F G A B C D E* *G A B C D E F*	*a*. *D G F C A B E* *F A B G D E C* *G E A B C D F*	*b*. *D G B C A E F* *F E A G D B C* *G A F B C D E*
c. *D G F B A E C* *F A B G C D E* *G E A C D B F*	*d*. *D E A G C B F* *F G B C A D E* *G A F B D E C*	*e*. *D A B G C E F* *F G A C D B E* *G E F B A D C*
f. *D A F G C B E* *F G A B D E C* *G E B C A D F*	*g*. *D G A B C E F* *F E B G A D C* *G A F C D B E*	

Table 3
Details of 7×7 Latin squares that can be partitioned into 4×7 and 3×7 Youden squares

Code Nos. of main classes (the numbering being that of Norton [1])	No. of isotopy classes per main class	No. of isotopy classes per main class that have partitionable members	Total no. of isotopy classes with partitionable members	Total No. of main classes with partitionable members
1	1	1	1	1
14	3	3	3	1
2, 7, 16, 33, 61, 71, 106, 121, 124	3	1	9	9
27, 112, 131	3	2	6	3
3, 31, 54, 93, 113	6	2	10	5
Total	—	—	29	19

columns has the same subset of the letters in each row. Within arrangements a to g in Table 2, each arrangement is obtained from the previous one by making the permutation $(ABCDEFG)$ of letters, followed by the permutation (1234567) of columns, and then — if necessary — reordering the rows.

The number of isotopy classes of 7×7 Latin squares obtained by appending one of the eight 3×7 Youden squares $X, a, b, \ldots, g$ from Table 2 to one of the six 4×7 Youden squares $1, \ldots, 6$ from Table 1 clearly cannot exceed $8 \times 6 = 48$ and is actually less. Table 3 reports that members of 29 isotopy classes of 7×7 Latin squares can be partitioned into 4×7 and 3×7 Youden squares, and that these 29 come from 19 main classes. These 19 do not include the single main class that Norton [1] failed to find [6], so his numbering of main classes can be used here.

If a Youden square has the kth letter in row i and column j, then its 'dual' [4] has the jth letter in row i and column k. Similarly, if a Latin square has the kth letter in row i and column j, we here define its 'column-letter adjugate' (CLA) to have the jth letter

Table 4
Representative 7×7 Latin squares that can be partitioned into 4×7 and 3×7 Youden squares[a]

mc1 {1}	A B C D E F G - - - - - - - - B C D E F G A C D E F G A B E F G A B C D - - - - - - - - D E F G A B C F G A B C D E G A B C D E F	mc14 {8}	A B C D E F G - - - - - - - - B D F C G E A D C E F A G B F E A G D B C - - - - - - - - C G D B F A E E F G A B C D G A B E C D F	mc14 (RC) + CLA {1 + 1}	A B C D E F G B D G C F E A D C B F A G E F E A G C B D - - - - - - - - C F D E G A B E G F A B D C G A E B D C F
mc2 (RL) {1}	A B C D E F G - - - - - - - - B G D A C E F C E F B G D A E F B G D A C - - - - - - - - D A E C F G B F C G E A B D G D A F B C E	mc7 {1}	D G B F C A E E F A B D G C F C D E G B A* G A F C B E D - - - - - - - - A B C D E F G B D E G A C F* C E G A F D B	mc16 {1}	D C B F G A E E F A B D G C F G D E C B A G A F C B E D - - - - - - - - A B C D E F G B D E G A C F C E G A F D B
mc33 {3}	B E A G D C F D A E F G B C E D F C A G B G C B E F A D - - - - - - - - A B C D E F G C F G A B D E F G D B C E A	mc61 {3}	B E D G F C A C G F E B A D D F B A C G E E C G B A D F - - - - - - - - A B C D E F G F D A C G E B G A E F D B C	mc71 {1}	A B C D E F G* B A E G D C F E D F C A G B G C B E F A D - - - - - - - - C F G A B D E D E A F G B C* F G D B C E A
mc106 {3}	A B C D E F G* B A E G F C D E D F C A G B G C B E D A F - - - - - - - - C F G A B D E D E A F G B C* F G D B C E A	mc121 {3}	A B C D E F G D F G E C B A E D F B A G C F C E G B A D - - - - - - - - B A D F G C E C G B A D E F G E A C F D B	mc124 {1}	C A E B D G F D F G C B E A E G D F A C B F D B G C A E - - - - - - - - A B C D E F G B E F A G D C G C A E F B D
mc27 (RC) + CLA {1 + 1}	A B C D E F G B E A G C D F C D G F B E A D G E C F A B - - - - - - - - E F D B A G C F A B E G C D G C F A D B E	mc112 (RC) + CLA {1 + 1}	A B C D E F G B D G F C E A C E A G B D F D G E C F A B - - - - - - - - E F D B A G C F A B E G C D G C F A D B E	mc131 (RC) + CLA {1 + 1}	A B C D E F G D E F A G C B E D G F B A C G C D B F E A - - - - - - - - B G E C A D F C F A G D B E F A B E C G D
mc3 (RL) + CLA {3 + 3}	E D B A F G C - - - - - - - - A B C D E F G B C F G A D E G E A C D B F - - - - - - - - C A D F G E B D F G E B C A F G E B C A D	mc31 (RL) + CLA {1 + 1}	C A E F D G B D F G C B A E E G D B A C F F D B G C E A - - - - - - - - A B C D E F G B E F A G D C G C A E F B D	mc54 (RL) + CLA {1 + 1}	C A E B D G F D F G C B A E E G D F A C B F D B G C E A - - - - - - - - A B C D E F G B E F A G D C G C A E F B D

Table 4. *Continued.*

mc93	*A B C D E F G*	mc113	*C A E F D G B*
(RL)	*D E A F C G B*	(RL)	*D F G C B E A*
	E D G B A C F		*E G D B A C F*
+ CLA	*F G D C B E A*	+ CLA	*F D B G C A E*
{1 + 1}	-----------------	{1 + 1}	-----------------
	B F E A G D C		*A B C D E F G*
	C A F G D B E		*B E F A G D C*
	G C B E F A D		*G C A E F B D*

[a]The main classes are taken in the same order as in Table 3, and the code number of each main class (mc) is again Norton's; for explanations of other notation and of the asterisks, see text.

in row i and column k. (The term 'parastrophe' is now often used in place of Norton's term 'adjugate'.) A Youden square and its dual may or may not belong to the same isotopy class, which is why a main class of Youden squares may contain 1 or 2 isotopy classes. A Latin square and its CLA may or may not belong to the same isotopy class, but must belong to the same main class; a main class of Latin squares may contain 1, 2, 3 or 6 isotopy classes. If certain rows of a Latin square constitute a Youden square, then so do the corresponding rows of its CLA. The second column of Table 3 shows that no 7×7 Latin square from a main class containing just 2 isotopy classes can be partitioned into 4×7 and 3×7 Youden squares.

Table 4 gives specimen members of all the 29 isotopy classes of Table 3, except that a specimen is omitted if its CLA is given; where there is such an omission, the corresponding printed square is marked ' + CLA'. Wherever possible, the specimens in Table 4 are as given by Norton [1], except that rows may have been reorderd to permit the partitioning. However, Norton [1] gave only one example for each main class of 7×7 Latin squares, so Table 4 has to include some Latin squares from isotopy classes not specifically illustrated by Norton. Any such Latin square in Table 4 is labelled (RC), to indicate a transpose of Norton's example (i.e. Rows have been interchanged with Columns), or (RL), to indicate interchange of Norton's Rows and Letters and then reordering of columns subsequent to the first, so as to obtain $AB \ldots G$ in that order in the row that comes first before any reordering needed for the partitioning.

For each isotopy class of Latin squares from Table 4, the number given in braces { } in Table 4 is the number of members of the isotopy class that are obtainable by adjoining a 3×7 Youden square from Table 2 to a 4×7 Youden square from Table 1. The numbers in braces thus sum to 48.

In four of the specimens in Table 4, there is a partitioning into a single row and two sets of 3 rows; each of the two sets of 3 rows constitutes a 3×7 Youden square, and the single row may be taken with either set to give a 4×7 Youden square. In three other specimens from Table 4, a row given in the 4×7 Youden square may be swapped with one in the 3×7 Youden square; the swappable rows are indicated by asterisks. Similar possible swaps of rows in the specimen from the highly symmetric main class 1 are many and obvious, and so are not indicated.

Table 5
Cross-partitioned 7×7 Latin squares from main classes containing 3 isotopy classes[a]

mc14
$$\begin{array}{cccc|ccc} A & G & E & C & F & D & B \\ G & C & F & D & E & B & A \\ E & F & G & B & D & A & C \\ C & D & B & E & A & G & F \\ \hline F & E & D & A & B & C & G \\ D & B & A & G & C & F & E \\ B & A & C & F & G & E & D \end{array}$$

mc27
$$\begin{array}{cccc|ccc} A & C & D & B & E & G & F \\ C & E & F & A & B & D & G \\ F & D & A & G & C & B & E \\ G & A & B & E & F & C & D \\ \hline E & G & C & D & A & F & B \\ B & F & G & C & D & E & A \\ D & B & E & F & G & A & C \end{array}$$

mc112
$$\begin{array}{cccc|ccc} A & C & D & B & E & G & F \\ C & A & F & E & B & D & G \\ F & D & A & G & C & B & E \\ G & E & B & A & F & C & D \\ \hline E & G & C & D & A & F & B \\ B & F & G & C & D & E & A \\ D & B & E & F & G & A & C \end{array}$$

mc131
$$\begin{array}{cccc|ccc} C & A & D & B & E & G & F \\ A & C & F & E & B & D & G \\ F & D & A & G & C & B & E \\ G & E & B & A & F & C & D \\ \hline E & G & C & D & A & F & B \\ B & F & G & C & D & E & A \\ D & B & E & F & G & A & C \end{array}$$

[a]The first example is symmetric, and each of the other three is invariant under transposition followed by the permutation $(BG)(DF)$.

If we take the CLA of any of the four Table 4 entries marked (RC), its columns can be reordered to produce a 'cross-partitioned' 7×7 Latin square partitioned by rows into 3×7 and 4×7 Youden squares and by columns into 7×3 and 7×4 Youden squares (i.e. 3×7 and 4×7 Youden squares rotated through 90 degrees). The four resultant cross-partitioned Latin squares are in Table 5, except that the rows and columns of the second, third and fourth examples have been further permuted, and the letters of the fourth example also permuted, in ways that make the very close relationships between the last three examples immediately clear on inspection.

3. Partitioning 7×7 Graeco-Latin squares

Norton [1, pp. 303–306] showed that there are seven main classes $\Gamma 1, \Gamma 2, \ldots, \Gamma 7$ of 7×7 Graeco-Latin squares, and that the Latin squares involved in these come from main classes $1, 2, 59, 70, 131$ and 146, as shown in Table 6. As three of these last main classes, namely $1, 2$ and 131 are represented in Tables 3 and 4 and are the only main classes to be involved in $\Gamma 4$, $\Gamma 5$, $\Gamma 6$ and $\Gamma 1$, we can therefore ask whether any members of $\Gamma 4$, $\Gamma 5$, $\Gamma 6$ and $\Gamma 1$ can be partitioned into 3×7 and 4×7 Graeco-Latin subsections consisting of one Youden square superimposed on another. We now show that such partitioning is indeed possible for $\Gamma 4$, $\Gamma 5$ and $\Gamma 1$ but not for $\Gamma 6$.

Table 6
The seven main classes of 7×7 Graeco-Latin squares and the main classes of Latin squares involved in them

Main class of Graeco-Latin squares	Main classes of the incorporated Latin squares
$\Gamma 1$	1, 2, 131, 131
$\Gamma 2$	1, 1, 1, 146
$\Gamma 3$	1, 59, 59, 59
$\Gamma 4$	1, 1, 1, 1
$\Gamma 5$	1, 1, 1, 1
$\Gamma 6$	2, 2, 2, 2
$\Gamma 7$	70, 70, 70, 70

After some reordering of rows and of columns, the Table 4 entry for Norton's main class 1 can be written in the following cross-partitioned 'self-orthogonal' form (i.e. orthogonal to its transpose):

$$\begin{array}{c|ccc|ccc}
A & C & E & B & F & D & G \\
\hdashline
B & D & F & C & G & E & A \\
C & E & G & D & A & F & B. \\
E & G & B & F & C & A & D \\
\hdashline
G & B & D & A & E & C & F \\
F & A & C & G & D & B & E \\
D & F & A & E & B & G & C
\end{array} \tag{3.1}$$

The Graeco-Latin square formed by superimposing (3.1) on its transpose, namely

$$\begin{array}{ccccccc}
AA & CB & EC & BE & FG & DF & GD \\
BC & DD & FE & CG & GB & EA & AF \\
CE & EF & GG & DB & AD & FC & BA \\
EB & GC & BD & FF & CA & AG & DE \\
GF & BG & DA & AC & EE & CD & FB \\
FD & AE & CF & GA & DC & BB & EG \\
DG & FA & AB & ED & BF & GE & CC
\end{array} \tag{3.2}$$

comes from $\Gamma 4$ [1, pp. 304–305]. By applying the permutation $(BEC)(DFG)$ to the letters of the first Latin square in (3.2) and then reordering rows and columns we can transform (3.2) to

$$\begin{array}{ccccccc}
AA & BB & CC & DD & EE & FF & GG \\
\hdashline
BE & CF & DG & EA & FB & GC & AD \\
CB & DC & ED & FE & GF & AG & BA \\
EC & FD & GE & AF & BG & CA & DB \\
\hdashline
DF & EG & FA & GB & AC & BD & CE \\
FG & GA & AB & BC & CD & DE & EF \\
GD & AE & BF & CG & DA & EB & FC
\end{array} \tag{3.3}$$

which is merely the Table 4 entry for Norton's main class 1 mated with the Latin square obtained from it by applying the permutation (234) (576) to the rows. Thus (3.3) is a Graeco-Latin square that can be partitioned into 3×7 and 4×7 superimpositions of one Youden square on another. Another such 7×7 Graeco-Latin square is

$$\begin{array}{ccccccc}
AA & BB & CC & DD & EE & EF & GG \\
\hdashline
BD & CE & DF & EG & FA & GB & AC \\
CG & DA & EB & FC & GD & AE & BF \\
EF & FG & GA & AB & BC & CD & DE \\
\hdashline
DC & ED & FE & GF & AG & BA & CB \\
FB & GC & AD & BE & CF & DG & EA \\
GE & AF & BG & CA & DB & EC & FD
\end{array}, \tag{3.4}$$

which comes from $\Gamma 5$, and is merely the Table 4 entry for main class 1 mated with the Latin square obtained from it by applying the permutation (264735) to the rows.

After some reordering of rows and of columns, the Table 4 entry for Norton's main class 2 can be written in the following self-orthogonal form similar to (3.1) but not cross-partitioned:

$$\begin{array}{c@{\quad}ccc@{\quad}ccc}
A & C & E & B & F & D & G \\
\hdashline
B & D & C & G & E & A & F \\
C & F & G & E & D & B & A \\
E & B & D & F & A & G & C \\
\hdashline
G & A & B & D & C & F & E \\
F & G & A & C & B & E & D \\
D & E & F & A & G & C & B
\end{array}. \tag{3.5}$$

The Graeco-Latin square formed by superimposing (3.5) on its transpose comes from $\Gamma 6$ [1, pp. 304–305] and cannot be partitioned to produce a 3×7 Graeco-Latin subsection where one 3×7 Youden square is superimposed on another. As only one of the three isotopy classes of Latin squares from main class 2 has members that can be partitioned into 3×7 and 4×7 Young squares, it follows from the 8-fold symmetry of $\Gamma 6$ [1, p. 305] that no Graeco-Latin square from $\Gamma 6$ can be partitioned to produce a 3×7 Graeco-Latin subsection obtainable by superimposing one 3×7 Youden square on another.

After some re-ordering of rows and of columns, and permutation of the letters, the Table 4 entry for Norton's main class 131 can be rewritten as either of the following orthogonal mates, the first three rows of either mate being the last three rows of the other:

$$
\begin{array}{c@{\quad}ccc@{\quad}ccc}
G & E & D & F & A & C & B \\
F & D & B & G & C & A & E \\
D & F & G & C & B & E & A \\
\\
A & C & E & B & D & G & F \\
\hdashline
B & A & F & E & G & D & C \\
C & B & A & D & E & F & G \\
E & G & C & A & F & B & D
\end{array}
\qquad
\begin{array}{c@{\quad}ccc@{\quad}ccc}
B & A & F & E & G & D & C \\
C & B & A & D & E & F & G \\
E & G & C & A & F & B & D \\
\hdashline
A & C & E & B & D & G & F \\
\\
G & E & D & F & A & C & B \\
F & D & B & G & C & A & E \\
D & F & G & C & B & E & A
\end{array}
\; . \qquad (3.6)
$$

Table 7
Two Graeco-Latin squares from $\Gamma 1$ that are partitionable between the fourth and fifth rows into two superimposed 4×7 Youden squares followed by two superimposed 3×7 Youden squares

Graeco-Latin square A:

$$
\begin{array}{ccc@{\quad}c|ccc}
C & A & D & B & E & G & F \\
A & C & F & E & B & D & G \\
F & D & A & G & C & B & E \\
G & E & B & A & F & C & D \\
\hdashline
E & G & C & D & A & F & B \\
B & F & G & C & D & E & A \\
D & B & E & F & G & A & C
\end{array}
\qquad\qquad
\begin{array}{ccc@{\quad}c@{\quad}ccc}
E & B & F & G & D & C & A \\
C & D & B & F & E & A & G \\
G & C & E & D & A & F & B \\
\hdashline
F & G & D & A & C & B & E \\
\hdashline
A & E & G & B & F & D & C \\
B & F & A & C & G & E & D \\
D & A & C & E & B & G & F
\end{array}
$$

Square from mc131 as in Table 5 — From square from mc2 as in Table 4

Graeco-Latin square B:

$$
\begin{array}{ccccccc}
A & B & C & D & E & F & G \\
\hdashline
B & C & D & E & F & G & A \\
C & D & E & F & G & A & B \\
E & F & G & A & B & C & D \\
\hdashline
D & E & F & G & A & B & C \\
F & G & A & B & C & D & E \\
G & A & B & C & D & E & F
\end{array}
\qquad\qquad
\begin{array}{ccccccc}
D & F & G & C & E & B & A \\
\\
A & C & B & F & G & D & E \\
B & E & A & D & C & F & G \\
C & A & F & G & B & E & D \\
\hdashline
G & D & E & B & A & C & F \\
F & G & C & E & D & A & B \\
E & B & D & A & F & G & C
\end{array}
$$

Square from mcl as in Table 4 — From square from mc131 as in Table 4

The Graeco-Latin square formed by superimposing one of the Latin squares (3.6) on the other comes from $\Gamma 1$ [1, pp. 303–304]. This Graeco-Latin square cannot be partitioned to produce a 3×7 or 4×7 Graeco-Latin superimposition of one Youden square on another. However, exhaustive examination of possibilities shows that such partitioning is possible, between the fourth and fifth rows, for the two Graeco-Latin squares from $\Gamma 1$ that are in Table 7.

Thus, in total, we have superimpositions of one 3×7 Youden square on another in the Graeco-Latin squares (3.3) and (3.4) and in the Graeco-Latin squares in Table 7. To compare these different superimpositions, we can, for any one of them, define the zero-one incidence matrix $\boldsymbol{n}$ to have its (i, j)th element as 1 $(i, j = 1, 2, \ldots, 7)$ if and only if the ith of the letters $A, B, \ldots, G$ from the first Youden square is paired, within the superimposition, with the jth of the letters from the second Youden square. It is then easy to see that, for each 3×7 superimposition in either (3.3) or (3.4), the matrix $\boldsymbol{n}$ is the incidence matrix of a symmetric balanced incomplete block design, whereas for each of the 3×7 superimpositions from Table 7, the matrix $\boldsymbol{n}$ is not the incidence matrix of a balanced incomplete block design.

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